The Adjunction Conjecture and its applications

Florin Ambro

Abstract

Adjunction formulas are fundamental tools in the classification theory of algebraic varieties. In this paper we discuss adjunction formulas for fiber spaces and embeddings, extending the known results along the lines of the Adjunction Conjecture, independently proposed by Y. Kawamata and V. V. Shokurov.

As an application, we simplify Kollár's proof for the Anghern and Siu's quadratic bound in the Fujita's Conjecture. We also connect adjunction and its precise inverse to the problem of building isolated log canonical singularities.

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Introduction

The rough classification of projective algebraic varieties attempts to divide them according to properties of their canonical class K_X. Therefore, whenever two varieties are closely related, it is essential to find formulas comparing their canonical divisors. Such formulas are called *adjunction formulas.* We first look at the following examples:

(1) Let C be a smooth curve on a smooth projective surface X. Then

$$(K_X + C)|_C \sim K_C$$

(2) Let $C \subset \mathbb{P}^2$ be the curve defined by the equation $x^2z - y^3 = 0$, with normalization $C^\nu \simeq \mathbb{P}^1$. Then

$$(K_{\mathbb{P}^2} + C)|_{C^\nu} \sim K_{C^\nu} + 2 \cdot P$$

where P is the point of C^ν above the cusp $(0 : 0 : 1) \in C$.

(3) Let X be a nonsingular variety and $W \subset X$ be the complete intersection of the nonsigular divisors $D_1, \dots , D_k$. Then

$$(K_X + \sum_i D_i)|_W \sim K_W$$

The appearance of the divisor $B_{C^\nu} = 2 \cdot P$, called the *different* of the log divisor $K_X + C$ on C^ν, is what makes the adjunction formula symmetric (the different happens to be trivial in the other two examples). This phenomena is called *subadjunction* in [KMM, 5-1-9] and it was first observed by Miles Reid in this context.

The examples above suggest that a *log divisor* $K_X + B$ has a natural *residue* $K_W + B_W$ on the normalization W of the intersection of components of B with coefficient 1. It is also expected that the *moduli part* [Ka2, Ka3] $M_W = (K_X + B)|_W - (K_W + B_W)$ is *semiample*, that is one of its multiples is base point free. These ideas are formalized in the following conjecture, independently proposed by Yujiro Kawamata and Vyachelsav Shokurov.

Conjecture 1 (The Adjunction Conjecture). *Let (X, B) be a log variety and let $j : W \to X$ be the normalization of an irreducible component of $LCS(X, B)$ such that $K_X + B$ is log canonical in the generic point of $j(W)$. There exists a canonically defined $\mathbb{R}$-divisor B_W on W, called the* different *of the log divisor $K + B$ on W, with the following properties:*

1. *(W, B_W) is a log variety, that is $K_W + B_W$ is $\mathbb{R}$-Cartier and B_W is an effective divisor;*
2. *The induced map $j : (W, B_W) \to (X, B)$ is* log proper*, i.e. for each closed subvariety $Z \subset W$, there exists a natural number $N \in \mathbb{N}$ such that*
$$\frac{a(j(Z); X, B)}{N} \le a(Z; W, B_W) \le a(j(Z); X, B)$$
3. *(Freeness) $(K + B)|_W \sim_{\mathbb{R}} K_W + B_W + M$, where M is an $\mathbb{R}$-free divisor on W;*
4. *(Boundness) If $K + B$ is $\mathbb{Q}$-Cartier of index r, there exists a natural number*
$$b = b(r, dimW, dimX) \in \mathbb{N}$$
such that $b(K_W + B_W)$ and bM are Cartier divisors.

Chapter 1 is introductory. We introduce in Chapter 2 Shokurov's *minimal log discrepancies* $a(Z;X,B)$, measuring the singularities of a log pair (X,B) in a closed subvariety Z, and we discuss two conjectures: the lower semicontinuity of minimal log discrepancies and the precise inverse of adjunction (the exceptional case).

Chapters 3 and 4 are an expanded version of Y. Kawamata's papers on adjunction [Ka1, Ka2, Ka3]. In Chapter 3 we define the *discriminant* B_Y of a log divisor $K_X + B$ along a morphism $f : X \to Y$. It measures the singularities of $K + B$ above the codimension 1 points of Y. The discriminant appears in [Ka2, Ka3] for special morphisms, as well as in [Mo, 5.12,9.12], where it is called the *negligible part*. For instance, a fiber space of smooth varieties $f : X \to Y$ with simple normal crossing ramification divisor is semistable in codimension 1 iff the discriminant of K_X is trivial.
We prove the finite base change formula for discriminants and we propose the Base Change Conjecture, claiming that the *birational* base change formula for discriminants holds for *log Calabi-Yau fiber spaces*. The Base Change Conjecture is intuitively equivalent to the log properness property in the Adjunction Conjecture. Finally, we present an extension of the result of Y. Kawamata on the nefness of the moduli part for certain log Calabi-Yau fiber spaces. The Base Change Conjecture implies that Kawamata's positivity result holds for every log Calabi-Yau fiber space.

In Chapter 4 we introduce the *different* of a log divisor $K_X + B$ on a log canonical (lc) center $W \subset X$ such that $a(\eta_W; B) = 0$. We restrict to the case when W is an *exceptional* lc center, the higher codimensional equivalent of generic pure log terminality. For instance, any codimension 1 lc center is exceptional. However, the definition and the properties of the different hold for non-exceptional lc centers too, once the basic adjunction calculus is extended from normal to seminormal varieties. The different is the discriminant of a log Calabi-Yau fiber space, so the properties of the latter translate into properties of the former. We also show that the Base Change Conjecture reduces the first two properties of the different stated in the Adjunction Conjecture to the case $codim(W,X) = 1$.

We conclude this chapter with an extension of Kawamata's adjunction formula [Ka3]. This weak version of adjunction is enough for certain applications. We use it in Chapter 5 to reobtain the known quadratic bound for building isolated log canonical singularites, found by U. Anghern and Y.T. Siu in the analytic case, later adapted by J. Kollár to the algebraic case.

Finally, we discuss applications of adjunction, as an excellent tool for inductive arguments in Higher Dimensional Algebraic Geometry. Chapter 5 deals with the problem of *building log canonical singularities*. If $x \in X$ is a closed point such that $a(x;B) \geq 0$ we search for effective $\mathbb{Q}$-divisors D such that $a(x;B+D) = 0$. To make the problem nontrivial, we fix an ample $\mathbb{Q}$-Cartier divisor H on X and ask what is the infimum $bld_x(B;H)$ of all $c > 0$ for which there exists a divisor $D \sim_{\mathbb{Q}} cH$ with the above property.

If X is a curve, then $bld_x(B;H) = a(x;B)/deg_X(H)$, in other words $bld_x(B;H)$ is controlled by the (global) numerical properties of H and the (local) invariants of the singularity of the log variety (X,B) at x. The optimal bound for $bld_x(B;H)$ is stated as Conjecture 7. We show that the first two properties stated in the Adjunction Conjecture reduce Conjecture 7 to its inductive step, stated as Conjecture 8. We also provide some evidence for Conjecture 8.
A lemma of Y. Kawamata translates any upper bound for $bld_x(B;H)$ in effective results on the global generation of (log) adjoint line bundles on projective varieties. In particular, the first two properties stated in the Adjunction Conjecture and Conjecture 8 imply a stronger version of Fujita's Conjecture.

As a final remark, Chapter 4 leaves the Adjunction Conjecture still in a hypothetical form. The only satisfactory case so far is $codim(j(W), X) = 1$, where all the properties are checked, with the exception of the *precise inverse of adjunction*. However, assuming that $j(W)$ is exceptional and that the Base Change Conjecture holds true we can summarize the known results as follows:

- Properties 1 and 2 in the Adjunction Conjecture hold, with the exception of precise inverse of adjunction, which is reduced to the divisorial case.

- The moduli part M_W is nef. Moreover, M_W is semiample if $codim(j(W), X) = 2$, according to [Ka2].

If W is a curve, we just need Finite Base Change (Theorem 3.2) instead of the Base Change Conjecture. Therefore the Adjunction Conjecture is proved if $dimX \leq 3$, with the exception of precise inverse of adjunction (boundness was basically proven by Y. Kawamata [Ka2]).

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1 The basics of log pairs

1.1 Prerequisites

A *variety* is a reduced irreducible scheme of finite type over a fixed field k, of characteristic 0.

Let X be a normal variety and K one of the rings $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. A K-*divisor* $B = \sum_i b_i B_i$ on X is a linear combination of prime Weil divisors with coefficients in K, i.e. an element of $N^1(X) \otimes K$. A K-divisor is said to be K-Cartier if it belongs to $Div(X) \otimes K \subset N^1(X) \otimes K$, where $Div(X)$ is the space of $\mathbb{Z}$-divisors which are Cartier. A K-divisor $B = \sum_i b_i B_i$ is *effective* if $b_i \geq 0$ for every i.

The fundamental invariant of X is its *canonical class* K_X. It is a $\mathbb{Z}$-Weil divisor, uniquely determined up to linear equivalence. In what follows, the choice of K_X in its class is irrelevant.

Two K-divisors D_1, D_2 are K-linearly equivalent, denoted by $D_1 \sim_K D_2$, if $D_1 - D_2$ belongs to $P(X) \otimes K \subset N^1(X) \otimes K$, where $P(X)$ is the group of principal $\mathbb{Z}$-divisors associated to nonzero rational functions on X.

A morphism $f : X \to Y$ is called a *contraction* if the natural morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism. Also, f is called an *extraction* if it is a proper birational morphism of normal varieties.

We say that a K-divisor D is K-*linearly trivial over* Y, denoted $D \sim_{K,f} 0$, if there exists a K-Cartier divisor D' on Y such that $D \sim_K f^*D'$. If f is a contraction, then the K-class of D' is uniquely determined by D. A K-Cartier divisor D on X is called f-*nef* if $D.C \geq 0$ for every proper curve C such that $f(C) =$ point.

1.2 Log varieties and pairs

The objects of the log-category are the singular counterpart of the smooth varieties with smooth boundary. They appear naturally in birational geometry.

Definition 1.1. *A* log pair (X, B) *is a normal variety X equipped with an $\mathbb{R}$-Weil divisor B such that $K + B$ is $\mathbb{R}$-Cartier. We will equivalently say that $K + B$ is a* log divisor. *A* log variety *is a log pair (X, B) such that B is effective. We call B the* pseudo-boundary *of the log pair.*

Definition 1.2.

1. *A log pair (X, B) has* log nonsigular support *if X is nonsingular and if $B = \sum b_i B_i$, then $\cup_{b_i \neq 0} B_i$ is a union of smooth divisors intersecting transversely (in other words, it has simple normal crossings).*
2. *A* log resolution *of a log pair (X, B) is an extraction $\mu : \tilde{X} \to X$ such that $\tilde{X}$ is nonsingular and* $\mathrm{Supp}(\mu^{-1}(B)) \cup Exc(\mu)$ *is a simple normal crossing divisor.*

One of the fundamental birational operations is the *pull back of log divisors.* If $\mu : \tilde{X} \to X$ is an extraction and $K + B$ is a log divisor on X, there exists a unique log divisor $K_{\tilde{X}} + B^{\tilde{X}}$ on $\tilde{X}$ such that

i) $B^{\tilde{X}} = \mu^{-1}B$ on $\tilde{X} \backslash Exc(\mu)$,

ii) $\mu^*(K + B) = K_{\tilde{X}} + B^{\tilde{X}}$.

The divisor $B^{\tilde{X}}$ is called the *log codiscrepancy divisor* of $K+B$ on $\tilde{X}$, making $(\tilde{X}, B^{\tilde{X}})$ a log pair which is identical to (X,B) from the singularities point of view. If $\mu : \tilde{X} \to (X,B)$ is a log resolution, then the log pair $(\tilde{X}, B^{\tilde{X}})$ has log nonsingular support. In the sequel, when we say that $\mu : (\tilde{X}, \tilde{B}) \to (X,B)$ is a log resolution, it is understood that $\tilde{B} = B^{\tilde{X}}$.

Example 1.1. *Let $\mu : \tilde{X} \to X$ be the blow-up of a subvariety W of X of codimension c, both nonsingular. Then $0^{\tilde{X}} = (1-c)E$ where E is the exceptional divisor. Therefore the log variety $X = (X, 0)$ is "similar" to the log pair $(\tilde{X}, (1-c)E)$. This illustrates the need for allowing the coefficients of the pseudo-boundary to take negative values.*

1.3 Singularities and log discrepancies

The class of log canonical singularities can be described as the largest class in which the LMMP seems to work, or as the smallest class containing Iitaka's log varieties which is closed under blow-ups.

Definition 1.3. *The log pair (X,B) has* log canonical singularities *(*lc *for short) if there exists a log resolution $\mu : (\tilde{X}, \tilde{B}) \to (X,B)$ such that all the coefficients of $\tilde{B}$ are at most* 1*.*

We say that (X,B) has Kawamata log terminal singularities *(*klt *for short) if there exists a log resolution $\mu : (\tilde{X}, \tilde{B}) \to (X,B)$ such that the coefficients of $\tilde{B}$ are all less than* 1*.*

It is easy to check that once $\tilde{B}$ has one of the above properties on a log resolution, it has the same property on any log resolution. In particular, a log pair (X,B) with log nonsingular support is log canonical (Kawamata log terminal) iff B has coefficients at most 1 (less than 1). Note that both classes of singularities defined above have local nature.

The singularities of log pairs are naturally described in terms of *log discrepancies*. Discrepancies are invariants attributed to Miles Reid who introduced them as means to control the canonical class of variety under a birational base change. A normalized version of discrepancies was also introduced in [Shif].

Definition 1.4. *Let (X,B) be a log pair. Let $E \subset Y \xrightarrow{\mu} X$ be a prime divisor on an* extraction *of X. The* log discrepancy *of E with respect to $K+B$ (or with respect to (X,B)), is defined as*

$$a_l(E; X, B) = 1 - e$$

where e is the coefficient of E in the log codiscrepancy divisor B^Y. By definition, $a_l(E;X,B) = 1$ if E is not in the support of B^Y. The center of E on X *is $\mu(E)$, denoted by $c_X(E)$.*
The log discrepancy $a_l(E;X,B)$ depends only on the discrete valuation defined by E on $k(X)$, in particular independent on the extraction Y where E appears as a divisor.
In this paper we will write $a(E;X,B)$ or $a(E;B)$, dropping the index l and even the variety X from the notation. However, $a(E;B)$ should not be confused with the standard notation in the literature for the discrepancy *of $K+B$ in E, which is equal to $-1 + a_l(E;X,B)$.*

Remark 1.1. *In the above notation, the log discrepancies for prime divisors on Y are uniquely determined by the formula*

$$\mu^*(K_X + B) = K_Y + \sum_{E \subset Y} (1 - a(E; X, B))E$$

where the sum runs over all prime divisors of Y.

Example 1.2. *Let (X, B) be a log pair with log nonsingular support and E the exceptional divisor on the blow-up of the nonsingular subvariety $Z \subset X$. Then*

$$a(E; X, B) = \operatorname{codim}(Z, X) - j + \sum_{j \in J} a(E_j; X, B),$$

where J is the set of components of B containing Z and $j = |J| \leq codim(Z, X)$. In particular, if $a(E_j; X, B) \geq 0$ for every $j \in J$, then

$$a(E; X, B) \geq \min_{j \in J} a(E_j; X, B) \geq 0$$

1.4 Log canonical centers

Let (X, B) be a log pair and $x \in X$ a closed point.

1. [Ka1] A *log canonical center* (*lc center* for short) of (X, B) is a closed subvariety $W \subset X$ such that $a(\eta_W; X, B) \leq 0$. The minimal element of the set
$$\{W'; x \in W', a(\eta_{W'}; X, B) \leq 0\}$$
if it exists, is called the *minimal lc center* at $x \in X$. If $\{x\}$ is not an lc center, then (X, B) is log canonical in a neighborhood of x, and moreover, the minimal lc center at x exists if there is an effective divisor $B^o \leq B$ such that $K_X + B^o$ is $\mathbb{R}$-Cartier and Kawamata log terminal in a neighborhood of x

2. (V. Shokurov) An lc center W is called *exceptional* if $a(\eta_W; B) = 0$ and on a log resolution $\mu : (\tilde{X}, \tilde{B}) \to (X, B)$ there exists a unique divisor E such that $W = c_X(E)$ and $a(E; B) \leq 0$ (in particular, $a(E; B) = 0$ if $dimW > 0$). The definition does not depend on the choice of the log resolution. It is the generic equivalent of pure log terminality.

3. We say that (X, B) has a *normalized minimal lc center at x* if $a(x; B) \geq 0$ and on a log resolution $\mu : (\tilde{X}, \tilde{B}) \to (X, B)$ there exists a unique divisor E such that $x \in c_X(E)$ and $a(E; B) \leq 0$. In particular, $a(E; B) = 0$ and $W = c_X(E)$ is the minimal lc center at x. The definition does not depend on the choice of the log resolution. Moreover, there is an open neighborhood U of x such that $LCS(X, B)|_U = W|_U$ as schemes. In particular, W is the only irreducible component of $LCS(X, B)$ passing through x.

If W is an lc center for (X, B), there might be several prime divisors E with $c_X(E) = W$ and $a(E; B) = 0$. Such divisors are called *lc places over* W [Ka1]. In fact, we have either infinitely many lc places over W, or exactly one. The latter holds precisely when W is an *exceptional lc center.*

The unique place is realized as a divisor on an extraction of X, and if $(E_1 \subset X_1 \to X)$ and $(E_2 \subset X_2 \to X)$ are two such realizations, then the induced birational morphism $\tau : X_1 \cdots > X_2$ sends E_1 onto E_2 and extends to an isomorphism in the generic point of E_1. All codimension 1 lc centers are exceptional (hopefully this does not cause any confusion).

Lemma 1.1. *[Ka1](Perturbation Lemma) Let $K_X + B^o$ and $K_X + B$ be two log divisors on X such that $0 \leq B^0 \leq B$ and $K_X + B^o$ is Kawamata log terminal in a neighborhood of x. If (X, B) is log canonical at x, with W the minimal lc center at x, there exists an effective $\mathbb{Q}$-Cartier divisor D such that $K_X + B^o + (1 - \epsilon)B + \epsilon D$ is log canonical with normalized minimal lc center W at x, for every $0 < \epsilon < 1$. Moreover, if H is a $\mathbb{Q}$-free divisor on X, we can assume $D \sim_{\mathbb{Q}} H$.*

1.5 The LCS locus

Let (X, B) be a log pair. The *locus of log canonical singularities* of $K_X + B$ [Sho4, 3.14] is the union of all lc centers:

$$LCS(X, B) = \bigcup_{\text{W lc center}} W$$

The name is slightly confusing, in the sense that W might be an lc center, without $K_X + B$ being log canonical in η_W. A correct notation, proposed by J. Kollár, is $Nklt(X, B)$: the abbreviation for the locus where (X, B) is not Kawamata log terminal. However, we will use Shokurov's notation since it better reflects its main use: to provide an induction step in higher dimensional algebraic geometry.

V. Shokurov also introduced a scheme structure on $LCS(X, B)$, defined as follows. Let $\mu : (\tilde{X}, \tilde{B}) \to (X, B)$ be a log resolution and P the truncation of $\tilde{B}$ to its components with coefficients at least 1. Then

$$\mathcal{I}(X, B) = \mu_* \mathcal{O}_{\tilde{X}}(\ulcorner -P \urcorner)$$

is a coherent ideal sheaf on X, independent of the choice of the log resolution. If B is effective, then $\mathcal{I}(X, B) \simeq \mu_* \mathcal{O}_{\tilde{X}}(\ulcorner -\tilde{B} \urcorner)$. The ideal sheaf $\mathcal{I}(X, B)$ defines a closed subscheme structure on $LCS(X, B) \subset X$.

The most general form of Kawamata-Viehweg vanishing theorem is:

Theorem 1.2. *(V. Shokurov) Let L be a Cartier divisor on a log variety (X, B) such that*

$$L \equiv K_X + B + H$$

where H is a nef and big $\mathbb{R}$-divisor. Then $H^j(X, \mathcal{I}(X, B) \otimes \mathcal{O}(L)) = 0$ for every $j > 0$. In particular, we have the following natural surjection

$$H^0(X, L) \to H^0(LCS(X, B), L|_{LCS(X,B)}) \to 0$$

The following simple result plays a crucial role in the inverse of adjunction:

Theorem 1.3. *(Connectedness Lemma [Sho4, 5.7], [Ko1, 17.4]) Let $\pi : X \to S$ be a contraction and $K_X + B$ a log divisor on X with the following properties:*

1. *$-(K_X + B)$ is π-nef and π-big,*
2. *the components of B with negative coefficients are π-exceptional.*

Then the induced map $LCS(X, B) \to S$ has connected fibers.

One of its applications is the following result of J. Kollár:

Proposition 1.4. *[Ko2, Corollary 7.8] Let $\{K_X + B_c\}_{c \in C}$ be an algebraic family of log divisors on X, parametrized by a smooth curve C. For each closed point $x \in X$, the following subset of C is closed:*

$$\{c \in C; x \in LCS(X, B_c)\} \subset C$$

2 Minimal log discrepancies

The minimal log discrepancy of a log pair (X, B) in a closed subvariety $W \subset X$ is an invariant introduced by V. Shokurov. It can be interpreted as the "dimension" of the singularity of (X, B) in W, although it distinguishes log canonical singularities only.

Definition 2.1. *(V.V. Shokurov) For a log pair (X, B) and a closed subset $W \subseteq X$ the following invariants are defined:*

- $a(W; X, B) = \inf\{a(E; X, B); \emptyset \neq c_X(E) \subseteq W\}$ *is called the minimal log discrepancy of (X, B) in W.*
- $a(\eta_W; X, B) = \inf\{a(E; X, B); c_X(E) = W\}$ *is called the minimal log discrepancy of (X, B) in the generic point of W.*

We have $a(\eta_W; X, B) \geq a(W; X, B)$ and strict inequality holds in general. In fact,

$$a(\eta_W; X, B) = a(W \cap U; U, B|_U)$$

for some generic open subset $U \subseteq X$ intersecting W. We abbreviate $a(X; X, B)$ and $a(\{x\}; X, B)$ by $a(X; B)$ and $a(x; B)$, respectively, where $x \in X$ is a closed point. The following lemma shows that the minimal log discrepancy is a well defined nonnegative real number if (X, B) is log canonical in a neighborhood of W, and is equal to $-\infty$ otherwise.

Lemma 2.1. *[Ko1, Proposition 17.1.1]*

1. *If (X, B) is not log canonical in a neighborhood of W, then*
$$a(W; X, B) = -\infty$$
2. *Assume that (X, B) is log canonical in a neighborhood of W. Let $(\tilde{X}, \tilde{B})$ be a log resolution of (X, B) such that $F_W = \mu^{-1}(W)_{red}$ is a divisor and* $\mathrm{Supp}(F_W) \cup \mathrm{Supp}(\tilde{B})$ *has simple normal crossings. Then $a(W; X, B) \in \mathbb{R}_{\geq 0}$ and*
$$\begin{aligned} a(W; X, B) &= \min\{a(F; X, B); F \text{ irreducible component of } F_W\} \\ &= \sup\{c \geq 0; (\tilde{X}, \tilde{B} + cF_W) \text{ is log canonical near } W\} \end{aligned}$$
Morever, the supremum is attained exactly on the components of F_W having log discrepancy minimal, that is equal to $a(W; X, B)$.

Proof. 1. If (X, B) is not log canonical in a neighborhood of W, there exists a prime divisor E on some extraction of X such that $a(E; B) < 0$ and $W \cap c_X(E) \neq \emptyset$. Let $x \in W \cap c_X(E)$ be a closed point. Suffices to show that $a(x; B) = -\infty$, since $a(W; B) \leq a(x; B)$. Let $\mu : (\tilde{X}, \tilde{B}) \to (X, B)$ be a log resolution such that E and $\mu^{-1}(x)$ are divisors on $\tilde{X}$. Since $x \in \mu(E)$, there exists a component E_0 of $\mu^{-1}(x)$ such that $E \cap E_0 \neq \emptyset$. Let X_1 be the blow up of $E \cap E_0$, with exceptional divisor E_1. Define inductively X_k to be the blow up of X_{k-1} in the intersection of E and E_{k-1}, with exceptional divisor E_k. An easy computation gives

$$a(E_k; B) = k \cdot a(E; B) + a(E_0; B), \quad c_X(E_k) = \{x\}$$

for every $k \in \mathbb{N}$. Therefore $\lim_{k \to \infty} a(E_k; B) = -\infty$, hence $a(x; B) = -\infty$.

2. Shrinking X to a neighborhood of W, we can assume that (X, B) is globally log canonical. It is enough to check the invariance of the minimum under blow-ups on $\tilde{X}$, which follows from Example 1.2.

□

2.1 The lower semicontinuity of minimal log discrepancies

Note what the first part of the proof of Lemma 2.1 actually says:

$$a(\eta_W; B) < 0 \Longrightarrow a(x; B) < 0 \text{ for every closed point } x \in W$$

This is typical for minimal log discrepancies: they are expected to behave in a lower semi-continuous fashion. To make this precise, fix a log pair (X, B) and consider the function

$$a : X \to \{-\infty\} \cup \mathbb{R}_{\geq 0},\ x \mapsto a(x; X, B)$$

Lemma 2.2. *The nonempty set $\{x \in X; a(x) \geq 0\}$ is the biggest open subset of X on which (X, B) has log canonical singularities. Its closed complement $\{x \in X; a(x) = -\infty\}$ is the union of all closed subvarieties W of X such that $a(\eta_W; B) = -\infty$.*

Proof. Let $x \in X$ such that $a(x) \geq 0$. According to the observation above, $a(\eta_W; B) \geq 0$ for every closed subvariety of X passing through x. Therefore there exists an open neighborhood V of x such that $(V, B|_V)$ is log canonical. In particular, $a(x') \geq 0$ for every $x' \in V$. Therefore $U = \{x \in X; a(x) \geq 0\}$ is open and (X, B) has log canonical singularities on U. The maximal property of U is clear and the next lemma shows that $U \neq \emptyset$. □

Lemma 2.3. *We have $a(x) = dimX$ if $x \in Reg(X) \setminus \mathrm{Supp}(B)$ and $a(x) < dimX$ if $x \in Reg(X) \cap \mathrm{Supp}(B)$ and B is effective. In particular, $a(x)$ is constant function equal to $dimX$ on an open dense subset of X.*

Proof. Let $x \in X$ a nonsingular point and let $E \subset \tilde{X} \to X$ be the exceptional divisor on the blow-up of x. Then $a(x; B) \leq a(E; B) = dimX - \mathrm{mult}_x(B) \leq dimX$, since B is effective. If $x \in Reg(X) \setminus \mathrm{Supp}(B)$, then $a(x; B) = a(E; B) = dimX$ according to Lemma 2.1.2. □

The following conjecture gives the effective upper bound for the function $a(x)$.

Conjecture 2. *(V.V. Shokurov [Sho3]) Let (X, B) be a log variety. Then*

$$\sup_{x \in X} a(x) = dimX$$

Moreover, the supremum is attained exactly on $Reg(X) \setminus \mathrm{Supp}(B)$.

The first part of Conjecture 2 can be reduced to the following conjecture.

Conjecture 3 (Lower semi-continuity). *Let (X, B) be a log variety. Then the function $a(x)$ is lower semi-continuous, i.e. every closed point $x \in X$ has a neighborhood $x \in U \subseteq X$ such that*

$$a(x; X, B) = \inf_{x' \in U} a(x'; X, B)$$

Indeed, the function $a(x)$ may jump only downwards in special points and it is constant equal to $dimX$ on an open dense subset of X. Therefore $\sup_{x \in X} a(x) = dimX$.

Example 2.1. *We check Conjecture 3 for $dimX \leq 2$. We may assume that (X, B) is log canonical near $P \in X$, since otherwise there is nothing to prove. In particular, B is effective with coefficients at most 1.*

a) Assume $\dim X = 1$*, and let* $B = \sum_j (1-a_j)P_j$*. Then*

$$a(x) = \begin{cases} 1 & \text{if } x \notin \{P_j\}_j \\ a_j & \text{if } x = P_j \end{cases}$$

Note that the effectivity of B *is essential for lower semi-continuity.*

b) Assume $\dim X = 2$*, and let* $B = \sum_j (1-a_j)B_j$*. We may assume that* $P \in \mathrm{Supp}(B_j)$ *for every* j*,* $U = X \setminus P$ *is nonsingular and* $B|_U$ *has nonsingular support. Then*

$$a(x) = \begin{cases} 2 & \text{if } x \in U \setminus \mathrm{Supp}(B) \\ 1 + a_j & \text{if } x \in U \cap B_j \end{cases}$$

Now we have two cases. If $P \in X$ *is a nonsingular point then* $a(P) \le \sum_j (a_j - 1) + 2 \le 1 + a_j$ *for every* j*. If* $P \in X$ *is singular, then it is well known that* $a(P) \le 1$ *and equality holds iff* $P \notin \mathrm{Supp}(B)$ *and* $(P \in X)$ *is a DuVal singularity (cf. [Al2, 3.1.2]).*

There are other interesting spectral properties of the minimal log discrepancies, conjectured by V. Shokurov, such as a.c.c. (see [Sho3, Ko1] for details). Minimal log discrepancies were successfully used by V. Shokurov for the existence and termination of log flips in dimension 3 [Sho4].

2.2 Precise inverse of adjunction

Conjecture 4. *(cf. [Ko1, Conjecture 17.3]) Let* (X, B) *be a log variety with a normalized minimal lc center* W *at* x*. Let* $\mu : (\tilde{X}, \tilde{B}) \to (X, B)$ *be a log resolution with* E *the only lc center above* W*. Set* $B_E = (\tilde{B} - E)|_E$*. Then*

$$a((\mu|_E)^{-1}(x); E, B_E) = a(x; X, B).$$

$$\begin{tikzcd} \tilde{X} \arrow[d, "\mu"'] & E \arrow[l, "\supset"'] \arrow[d, "\mu"] \\ X & W \arrow[l, "\supset"] \end{tikzcd}$$

First of all, it is clear that $a((\mu|_E)^{-1}(x); E, B_E) \ge a(x; X, B)$, so we just have to prove the opposite inequality:

$$a(E \cap \mu^{-1}(x); E, B_E) \le a(x; X, B)$$

According to [Ko1, Chapter 17], this inequality is implied by the Log Minimal Model Program in the case $0 \le a(x; X, B) \le 1$.

We describe in the next lemma a naive approach. Let $\tilde{B} = E + A$, hence $B_E = A|_E$. Assume $a(x; X, B) \ge 0$, $F_x = (\mu^{-1}(x))_{red}$ is a divisor and $\mathrm{Supp}(\tilde{B}) \cup \mu^{-1}(x)$ has simple normal crossings.

Lemma 2.4. *With the above notations, assume moreover there exists an effective* $\mathbb{R}$*-divisor* $\tilde{F}_x$ *supported in* $\mu^{-1}(x)$ *with the following properties:*

1. $-\tilde{F}_x$ *is* μ*-nef,*

2. *the supremum* $a = \sup\{\alpha \geq 0; K_{\tilde{X}} + \tilde{B} + \alpha\tilde{F}_x$ *is log canonical above* $x\}$ *is attained only on components* F *with* $a(F;B) = a(x;B)$.

Then $a((\mu|_E)^{-1}(x); E, B_E) \leq a(x; X, B)$.

Proof. Since $-(K_{\tilde{X}} + \tilde{B} + a\tilde{F}_x) = -\mu^*(K_X + B) - a\tilde{F}_x \equiv_\mu -a\tilde{F}_x$ is μ-nef, the induced map

$$LCS(\tilde{X}, \tilde{B} + a\tilde{F}_x) \to X$$

has connected fibers from the Connectedness Lemma. The only candidates for components of $LCS(\tilde{X}, \tilde{B}+a\tilde{F}_x)$ are E and components of F_x where a is attained. Therefore there exists a component F of F_x such that $F \cap E \neq \emptyset$ and $a(F;B) = a(x;B)$. Finally, $a(F \cap E; E, B_E) = a(F; \tilde{X}, \tilde{B}) = a(x; X, B)$, hence the desired inequality. □

Note that condition 2.4.2 is implied by the following

$$(2') \quad a(F_1;B) > a(F_2;B) \Longrightarrow \frac{a(F_1;B)}{\mathrm{mult}_{F_1}(\tilde{E}_x)} > \frac{a(F_2;B)}{\mathrm{mult}_{F_2}(\tilde{E}_x)}$$

for any two components F_1, F_2 of F_x. In particular, F_x has this property. Unfortunately, $-F_x$ is not μ-nef in general.

Example 2.2. *If* $x \in Reg(X) \setminus Supp(B)$ *and* μ *is the blow up of* $x \in X$, *with exceptional divisor* E, *then* $\tilde{F}_x = E$ *satisfies the assumptions of the lemma.*

The following partial result on inverse of adjunction is due to V. V. Shokurov [Sho4, 3.2] in the case $dim X = 3$. János Kollár later found a formal proof based on the Connectedness Lemma, which also proves the following result.

Theorem 2.5. *(cf. [Ko1, Theorems 17.6,17.7]) Let* (X, B) *be a log variety,* W *an exceptional lc center with lc place* E *and* $x \in W$ *a closed point.*

1. *W is the minimal lc center at x for (X,B) iff (E, B_E) is Kawamata log terminal over a neighborhood of x in X.*

2. *Assume that W is the minimal lc center at x for (X,B) and let D be an effective $\mathbb{R}$-Cartier divisor on X whose support does not contain W. Then $(X, B+D)$ is log canonical at x iff $(E, (B+D)_E)$ is log canonical above x.*

Proof. 1. Assume that E is realized as a divisor on the log resolution $\mu : (\tilde{X}, \tilde{B}) \to (X, B)$, with $\tilde{B} = E + A$.

$$\begin{tikzcd} \tilde{X} \arrow[d, "\mu"'] & E \arrow[l, "\supset"'] \arrow[d, "\mu"] \\ X & W \arrow[l, "\supset"] \end{tikzcd}$$

Note first the following equivalences:

a) (E, B_E) is Kawamata log terminal over a neighborhood of x in X iff $\mu^{-1}(x) \cap E \cap A^{\geq 1} = \emptyset$.

b) W is the minimal lc center at x for (X, B) iff $\mu^{-1}(x) \cap A^{\geq 1} = \emptyset$.

Therefore the implication "$\Rightarrow$" is clear. For the converse, assume that $\mu^{-1}(x) \cap E \cap A^{\geq 1} = \emptyset$. Then $\mu^{-1}(x) \cap E \neq \emptyset$ and $\mu^{-1}(x) \cap A^{\geq 1}$ partition the set $\mu^{-1}(x) \cap (E \cup \mathrm{Supp}(A^{\geq 1}))$.
But $E \cup \mathrm{Supp}(A^{\geq 1}) = LCS(\tilde{X}, \tilde{B})$ and since $-(K_{\tilde{X}} + \tilde{B}) \equiv_\mu 0$ and $\mu_* \tilde{B} = B$ is effective, the Connectedness Lemma implies that $\mu^{-1}(x) \cap LCS(\tilde{X}, \tilde{B})$ is a connected set. Therefore $\mu^{-1}(x) \cap A^{\geq 1} = \emptyset$.

2. The implication "$\Rightarrow$" is clear in *ii*), so we only prove the converse. Assume that $(E, (B+D)_E)$ is log canonical above x. Shrinking X near x, we can assume $(E, (B+D)_E)$ is globally log canonical and (E, B_E) is Kawamata log terminal. Since $(B+D)_E = B_E + \mu^*(D|_W)$ and D is effective, $(E, (B+tD)_E)$ is Kawamata log terminal for every $t < 1$. From *i*), W is the minimal lc center at x for $(X, B+tD)$ for every $t < 1$. In particular $a(x; B+tD) \geq 0$ for every $t < 1$, hence for $t = 1$ too.

$\square$

3 Adjunction for fiber spaces

Let $f : X \to Y$ be a proper contraction of normal varieties and $K_X + B$ a log divisor which is log canonical over the generic point of Y. We first introduce B_Y, the *discriminant* of the log divisor $K_X + B$ on Y. If $K_Y + B_Y$ is $\mathbb{R}$-Cartier, we could say that $K_Y + B_Y$ is the divisorial push-forward of the log divisor $K_X + B$.

Restricting afterwards our attention to the case when $K_X + B \sim_{\mathbb{R},f} 0$ and $K_Y + B_Y$ is a log divisor, we will study

a) the relation between the singularities of (X, B) and (Y, B_Y);

b) the positivity properties of the $\mathbb{R}$-class $M \in \mathrm{Pic}(Y) \otimes \mathbb{R}$ uniquely defined by the adjunction formula
$$K_X + B - f^*(K_Y + B_Y) \sim_{\mathbb{R}} f^*(M)$$

We say that $K_Y + B_Y + M$ is the *push-forward* of the log divisor $K_X + B$ on Y, being a combination of the *divisorial part* $K_Y + B_Y$ and the *moduli part* M.

Naturally, the push forward should be the inverse of the pull back operation. The latter is naturally defined when $f : X \to Y$ is a finite or birational morphism: if $K_Y + B$ is a log divisor on Y, there exists an induced log divisor $K_X + B^X$ on X uniquely defined by the adjunction formula
$$K_X + B^X = f^*(K_Y + B)$$
Then $(B^X)_Y = B$ and $M = 0$, that is B is the discriminant of $K_X + B^X$ on Y and the moduli part is trivial.

3.1 The discriminant of a log divisor

The following is the invariant form of the definition proposed by Y. Kawamata in [Ka2, Ka3].

Definition 3.1. *Let $f : X \to Y$ be a surjective morphism of normal varieties and $K_X + B$ a log divisor which is log canonical over the generic point of Y. For a prime divisor $Q \subset Y$ define*
$$a_Q = \sup\{c \in \mathbb{R}; K_X + B + cf^*Q \text{ is log canonical over } \eta_Q\}$$
Then $B_Y = \sum_Q (1 - a_Q)Q$ is a well defined $\mathbb{R}$-Weil divisor on Y, called the discriminant *of the log divisor $K_X + B$ on Y.*

Remark 3.1. 1. *By abuse of language, f^*Q is defined as the divisor associated to the pullback f^*t of a local parameter t of Q on Y. Since the supremum is defined over the generic point of Q, the choice of t is irrelevant.*

2. *If $f' : (X', B' = B^{X'}) \to (X, B) \to Y$ is the map induced by a crepant extraction or a finite cover of (X, B), then $B'_Y = B_Y$. In other words, for computing B_Y we are free to replace (X, B) by any crepant extraction or finite cover $(X', B^{X'})$.*

3. *B_Y is well defined since $a_Q = 1$ for all but a finite number of prime divisors. Indeed, assuming that X is nonsingular, there exists a non-empty open subset $U \subseteq Y$ such that $K_X + B$ is log canonical over U and f has nonsingular (possibly disconnected) fibers over U. This implies $a_Q = 1$ for every prime divisor Q with $Q \cap U \neq \emptyset$.*

4. *For any prime divisor Q, a_Q is a real number because $K_X + B$ is log canonical over η_Y. To compute a_Q, we can assume after blowing up X that $(X, B_{red} + (f^*Q)_{red})$ is log nonsingular over an open subset $U \subseteq Y$ with $U \cap Q \neq \emptyset$. Let $f^*Q = \sum_j w_j P_j$ over η_Q (note that $f(P_j) = Q$ for every P_j). Then*

$$a_Q = \min_j \frac{a(P_j; B)}{w_j}$$

In other words, if $b_j = \mathrm{mult}_{P_j}(B)$ for every j, then

$$b_Q = 1 - a_Q = \max_j \frac{b_j + w_j - 1}{w_j}$$

This is exactly the formula proposed in [Ka2, Ka3]. In particular, B_Y has rational coefficients if B does.

5. *In the above notation, $\frac{1}{N} \min_j a(P_j; B) \leq a_Q \leq \min_j a(P_j; B)$, where $N = \max_j w_j \in \mathbb{N}$. The presence of N makes the precise inverse of adjunction for higher codimension lc centers an inequality instead of equality.*

6. *(Additivity) If D is an $\mathbb{R}$-Cartier divisor on Y, then $K_X + B + f^*D$ is again log canonical over η_Y and $(B + f^*D)_Y = B_Y + D$.*

Example 3.1. 1. *Assume that f is birational and $K_X + B \sim_{\mathbb{R},f} 0$. Then $B_Y = f_*(B)$, $K_Y + B_Y$ is $\mathbb{R}$-Cartier and $f : (X, B) \to (Y, B_Y)$ becomes a crepant extraction, that is $K_X + B = f^*(K_Y + B_Y)$.*

2. *Assume that f is a finite map and $K_X + B^X$ is the pull back of the log divisor $K_Y + B$. Then*

$$(B^X)_Y = B$$

Moreover, if P is a prime divisor on X, $Q = f(P)$ and $w = \mathrm{mult}_P(f^*Q)$, then $a_Q = a(Q; Y, B) = a(P; X, B^X)/w$.

3. *Assume $f : X \to Y$ is a fiber space of smooth varieties with simple normal crossing ramification. Then f is semistable in codimension 1 iff $K_X = K_X + 0$ has trivial discriminant on Y.*

4. *Assume that $Y = C$ is a smooth curve. Then $f : (X, B) \to C$ is log canonical in the sense of [KM, Definition 7.1] iff (C, B_C) has canonical singularities, that is $B_C \leq 0$.*

The following result of Y. Kawamata gives a cohomological sufficient condition for the effectivity of the discriminant.

Lemma 3.1. *[Ka3] Let $f : X \to Y$ be a surjective map and $K_X + B$ a log divisor with log nonsingular support which is Kawamata log terminal over η_Y. Let Q be prime divisor on Y such that the coefficient of Q in the discriminant B_Y is negative. Then the following hold:*

- *$\ulcorner -B \urcorner$ is effective over the generic point of Q.*

- *The induced map $\mathcal{O}_{Y,Q} \to (f_* \mathcal{O}_X(\ulcorner -B \urcorner))_Q$ is not surjective.*

Proof. Since $\mu_*\mathcal{O}_{\tilde{X}}(\ulcorner -\tilde{B}\urcorner) = \mathcal{O}_X(\ulcorner -B\urcorner)$ for any log resolution $\mu : (\tilde{X}, \tilde{B}) \to (X, B)$, we can assume that we are in the situation and notation of Remark 3.1.4. Then $b_Q < 0$ is equivalent to

$$\ulcorner -b_j \urcorner \geq w_j \text{ for every } j$$

Since $\ulcorner -B^h \urcorner \geq 0$, there exists an open subset $U \subset X$ such that $U \cap Q \neq \emptyset$ and $\ulcorner -B\urcorner$ is effective on $\mu^{-1}(U)$. Moreover, the induced inclusion $\mathcal{O}_{Y,Q} \to (f_*\mathcal{O}_X(\ulcorner -B\urcorner))_Q$ factors as:

$$\mathcal{O}_{Y,Q} \to \mathcal{O}_Y(Q)_Q \to f_*\mathcal{O}_X(\ulcorner -B\urcorner)_Q$$

In particular, the natural inclusion $\mathcal{O}_{Y,Q} \to (f_*\mathcal{O}_X(\ulcorner -B\urcorner))_Q$ cannot be surjective. □

3.2 Base change for the divisorial push forward

The following result shows that the divisorial push forward of a log divisor commutes with finite base changes.

Theorem 3.2 (Finite Base Change). *Let $f : X \to Y$ be a proper surjective morphism of normal varieties and $K_X + B$ a K-Cartier log divisor which is log canonical over η_Y. Let $f : X' \to Y'$ be a morphism induced by the finite base change $\sigma : Y' \to Y$, and set $B' = B^{X'}$.*

$$\begin{tikzcd} X & X' \arrow[l, "\nu"'] \arrow[d, "f'"] \\ Y \arrow[from=u, "f"'] & Y' \arrow[l, "\sigma"] \end{tikzcd}$$

Then $K_{X'} + B'$ is log canonical over η_Y and $\sigma^*(K_Y + B_Y) = K_{Y'} + (B')_{Y'}$ as K-Weil divisors. Moreover, $K_Y + B_Y$ is K-Cartier iff $K_{Y'} + (B')_{Y'}$ is K-Cartier.

Proof. To check the equality $\sigma^*(K_Y + B_Y) = K_{Y'} + (B')_{Y'}$ we choose an arbitrary prime divisor $Q' \subset Y'$, with $\sigma(Q') = Q$ and $\mathrm{mult}_{Q'}(\sigma^*Q) = w$. From Example 3.1.2, we have to show that $a_{Q'} = w \cdot a_Q$.
If $c \leq a_Q$, then $K_X + B_X + cf^*Q$ is log canonical over η_Q, hence $K_{X'} + B' + c(f \circ \nu)^*Q = K_{X'} + B' + cf'^*\sigma^*Q$ is log canonical over η_Q. But $K_{X'} + B' + cf'^*\sigma^*Q \geq K_{X'} + B' + cwf'^*Q'$, so $K_{X'} + B' + cwf'^*Q'$ is log canonical. Hence $cw \leq a_{Q'}$. In particular, $a_{Q'} \geq w \cdot a_Q$.
Conversely, let $c \geq a_Q$. After possible blow-ups on X, there exists a prime divisor P on X with $a(P; B + cf^*Q) \leq 0$ and $f(P) = Q$. Since X' is a resolution of $X \times_Y Y'$, there exists a prime divisor P' on X' with $\nu(P') = P, f'(P') = Q'$. By Example 3.1.2, $a(P'; B' + c \cdot wf'^*Q') = a(P'; B' + c(f \circ \sigma)^*Q) \leq 0$, so $c \cdot w \geq a_{Q'}$. Therefore $w \cdot a_Q \geq a_{Q'}$.
The rest follows from the next lemma. □

Lemma 3.3. *Let $\sigma : Y' \to Y$ be a finite map of normal varieties, D a $\mathbb{Q}$-Weil divisor on Y, $D' = \sigma^*(D)$ the pull back of D, which is a $\mathbb{Q}$-Weil divisor on Y', and $r \in \mathbb{N}$. Then*

1. *if rD is Cartier, then rD' is Cartier;*

2. *if rD' is Cartier then $(deg(\sigma) \cdot r)D$ is Cartier.*

Under certain conditions, we expect that the divisorial push forward commutes with birational base changes too. According to [Mo, 5.12,9.12] and [Ka2, Ka3], we anticipate the following conjecture to be true. A partial result in this direction is Proposition 3.6.

Conjecture 5 (The Base Change Conjecture). *Let $f : X \to Y$ be a contraction of normal varieties and let $K_X + B$ be log divisor with the following properties:*

- *$K_X + B \sim_{\mathbb{Q},f} 0$;*
- *$K_X + B$ is Kawamata log terminal over η_Y;*
- *(X, B) has log nonsingular support and $\mathcal{O}_{Y,\eta_Y} = (f_* \mathcal{O}_X(\ulcorner -B \urcorner))_{\eta_Y}$.*

Then $K_Y + B_Y$ is $\mathbb{Q}$-Cartier and if $f' : (X', B^{X'}) \to Y'$ is a contraction induced by a birational base change $\sigma : Y' \to Y$, then $(B_Y)^{Y'} = (B^{X'})_{Y'}$. In other words,

$$\sigma^*(K_Y + B_Y) = K_{Y'} + (B^{X'})_{Y'}.$$

The divisor $K_Y + B_Y$ is always $\mathbb{Q}$-Cartier if Y is $\mathbb{Q}$-factorial, in particular nonsingular. As for the base change, even if it does not hold for $f : (X, B) \to Y$, it should hold for data $f' : (X', B') \to Y'$ induced on "sufficiently large extractions" of Y.

The Base Change Conjecture is intuitively equivalent to the Inverse of Adjunction Conjecture. As the next result shows, the log divisor $K_X + B$ and its divisorial push forward log divisor should be *in the same class of singularities.*

Proposition 3.4. *Assume the Base Change Conjecture holds true for $f : (X, B) \to Y$ and let Z be a closed proper subset of Y. There exists a positive natural number $N \in \mathbb{N}$ such that*

$$\frac{1}{N} a(f^{-1}(Z); X, B) \le a(Z; Y, B_Y) \le a(f^{-1}(Z); X, B).$$

Proof. There exists a fiber space $f' : (X', B') \to Y'$ induced by a birational base change $\sigma : Y' \to Y$ with the following properties:

- Y' is nonsingular, $\sigma^{-1}(Z)$ is a divisor and $\operatorname{Supp}(\sigma^{-1}(Z)) \cup \operatorname{Supp}((B_Y)^{Y'})$ is included in a snc divisor $Q = \sum_l Q_l$;
- X' is nonsingular and $\operatorname{Supp}(B') \cup \operatorname{Supp}(f'^* Q)$ is included in a snc divisor $P = \sum_j P_j$;
- there exists an index j_0 such that $P_{j_0} \subseteq (\sigma \circ f')^{-1}(Z)$, $f'(P_{j_0}) = Q_{l_0}$ for some index l_0 and $a(P_{j_0}; X, B) = a(f^{-1}(Z); X, B)$.

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "\nu"'] \arrow[d, "f'"] \\ Y & Y' \arrow[l, "\sigma"] \end{tikzcd}$$

Indeed, the first two property are obtained by letting $\sigma : Y' \to Y$, and then $X' \to X \times_Y Y'$ be "large enough" resolutions. As for the third, let P_{j_0} included in $f'^{-1}\sigma^{-1}(Z)$ such that $a(P_{j_0}; X, B) = a(f^{-1}(Z); X, B)$. If $f'(P_{j_0})$ is a divisor, there is some l_0 with $f'(P_{j_0}) = Q_{l_0}$, so we are done. Otherwise, by further blow-ups on Y' and X' we can assume

the proper transform of $P_{l'}$ maps to a divisor. Note that we do not change any horizontal component, since we only perform operations over proper subsets of Y.

Then $a(Q_{l_0}; B_Y) = a(Q_{l_0}; B_{Y'}) \leq a(P_{j_0}; B') = a(f^{-1}(Z); B)$, hence

$$a(Z; B_Y) \leq a(f^{-1}(Z); B)$$

On the other hand, if $N = \max\{w_{lj}\} \in \mathbb{N}$ and Q_l is any divisor of Q contained in $\sigma^{-1}(Z)$, then $a(Q_l; B_Y) = a(Q_l; B_{Y'}) \geq \frac{1}{N} \min_{f'(P_j)=Q_l} a(P_j; B') \geq \frac{1}{N} a(f^{-1}(Z); B)$. Taking infimum after all these Q_l's we obtain the other inequality. □

3.3 Positivity of the moduli part

Let $f : (X, B) \to Y$ be a data satisying the assumptions of the Base Change Conjecture. Assuming that $K_Y + B_Y$ is $\mathbb{Q}$-Cartier, there exists a unique class $M_Y \in \mathrm{Pic}(Y) \otimes \mathbb{Q}$ satisying the following adjunction formula:

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y + M_Y).$$

We can rewrite the above formula as $K_{(X,B)/(Y,B_Y)} := K_X + B - f^*(K_Y + B_Y) \sim_{\mathbb{Q}} f^*M$. Thinking of $f : (X, B) \to (Y, B_Y)$ as being the log analogue of semistable in codimension 1 morphisms, the line bundle $\mathcal{O}_Y(\nu M)$ corresponds to $f_*\mathcal{O}_X(\nu K_{(X,B)/(Y,B_Y)})$ for divisible and large enough integers $\nu \in \mathbb{N}$. Therefore we expect the following conjecture on the positivity of log-Hodge bundles to be true.

Conjecture 6 (Positivity). *(cf. [Ka2, Ka3],[Mo, 5.12,9.12]) Let $f : X \to Y$ be a contraction of normal varieties and let $K_X + B$ be log divisor with the following properties:*

- $K_X + B \sim_{\mathbb{Q},f} 0$;
- $K_X + B$ *is Kawamata log terminal over* η_Y;
- (X, B) *has log nonsingular support and* $\mathcal{O}_{Y,\eta_Y} = (f_*\mathcal{O}_X(\ulcorner -B \urcorner))_{\eta_Y}$.

Then $K_Y + B_Y$ is $\mathbb{Q}$-Cartier and the moduli part M_Y is $\mathbb{Q}$-free, that is the line bundle $\mathcal{O}_Y(\nu M_Y)$ is generated by global sections for some integer $\nu \in \mathbb{N}$.

Remark 3.2. *The connectivity assumption on the fibers of f is essential for the positivity of M. For instance, assume $f : X \to Y$ is a finite morphism between two smooth projective curves and let $K_X + B$ be a log divisor on X. Then $K_Y + B_Y$ is a log divisor and*

$$K_X + B_X - f^*(K_Y + B_Y) = -\Delta$$

where Δ is an effective divisor whose support does not contain any set theoretic fiber of f. Therefore $-M$ is nef. An important particular case is when f is a Galois cover and K_X+B is Galois invariant. Then $M = 0$.

Under an extra assumption, Y. Kawamata proved that the moduli part M_Y is a nef divisor [Ka3], and moreover, M_Y is $\mathbb{Q}$-free if f has relative dimension 1 [Ka2]. We end this section with his positivity result.

Theorem 3.5. *(cf. [Ka3, Theorem 2]) Consider the following setting:*

1. *$f : X \to Y$ is a contraction of nonsingular projective varieties;*

2. *$K_X + B \sim_{\mathbb{Q},f} 0$;*

3. *there exist simple normal crossing divisors $P = \sum P_j$ and $Q = \sum Q_l$ on X and Y, respectively, such that $\mathrm{Supp}(B) \subset P$, $f^{-1}(Q) \subset P$ and f is smooth over $Y \backslash Q$;*

4. *$B = B^h + B^v$ such that any irreducible component of B^h is mapped surjectively onto Y by f, $f : Supp(B^h) \to Y$ is relatively normal crossing over $Y \setminus Q$, and $f(Supp(B^v)) \subset Q$. An irreducible component of B^h (resp. B^v) is called horizontal (resp. vertical);*

5. *$K_X + B$ is Kawamata log terminal over η_Y and $\mathcal{O}_{Y,\eta_Y} = (f_*\mathcal{O}_X(\ulcorner -B \urcorner))_{\eta_Y}$.*

Then the moduli part M_Y is a nef divisor on Y.

Remark 3.3. *In [Ka3, Theorem 2] it is further assumed that $\ulcorner -B \urcorner$ is effective (that is (X, B) has Kawamata log terminal singularities), although this assumption is not used in the proof. Indeed, set $D = B - f^*B_Y$ and*

$$Z = \bigcup \{f(P_j); P_j \subset \mathrm{Supp}(B), codim(f(P_j), Y) \geq 2\}.$$

The following hold:

- *$K_X + D \sim_{\mathbb{Q}} f^*(K_Y + M_Y)$;*
- *$D_Y = 0$;*
- *$\ulcorner -D \urcorner$ is effective over $Y \backslash Z$;*
- *$Z \subset Q$ and $\mathrm{codim}(Z, Y) \geq 2$.*

Including Z in the closed subset of codimension at least 2 that is disregarded throughout the proof of [Ka3, Theorem 2], we obtain the nefness of M_Y.

We say that $f : (X, B) \to Y$ has the property $(\star)$ if the assumptions of the above theorem hold true.

Proposition 3.6. *Let $f : (X, B) \to Y$ and $f' : (X', B') \to Y'$ be fiber spaces satisfying the property $(\star)$ such that the following hold:*

1. *f' is induced by f by the birational base change σ, $\sigma^{-1}(Q) \subset Q'$ and $\nu^{-1}(P) \subset P'$;*
2. *$B' = B^{X'}$.*

$$\begin{tikzcd}
(X, B) \arrow[d, "f"'] & (X', B') \arrow[l, "\nu"'] \arrow[d, "f'"] \\
Y & Y' \arrow[l, "\sigma"]
\end{tikzcd}$$

Then $\Sigma = (B^{X'})_{Y'} - (B_Y)^{Y'}$ is an effective σ-exceptional divisor.

Proof. We have $\nu^*(K_X + B) = K_{X'} + B' \sim_{\mathbb{Q}} f'^*(K_{Y'} + (B')_{Y'} + M')$. On the other hand, $\nu^*(K_X + B) \sim_{\mathbb{Q}} \nu^* f^*(K_Y + B_Y + M) = f'^* \sigma^*(K_Y + B_Y + M)$. Since f' is a contraction, we infer that

$$\sigma^*(K_Y + B_Y + M) \sim_{\mathbb{Q}} K_{Y'} + (B')_{Y'} + M'$$

Therefore $\Sigma \sim_{\mathbb{Q}} -M' + \sigma^* M$. Clearly Σ is σ-exceptional. Moreover, since $-\Sigma$ is σ-nef, the negativity of the birational contraction σ implies that Σ is effective. $\square$

4 Adjunction on log canonical centers

Let $K+B$ be a log divisor on a normal variety X. For a closed subvariety $W \subset X$ such that $a(\eta_W; X, B) = 0$, the Adjunction Conjecture predicts that the different B_{W^ν} induced on the normalization of W has the following properties:

- $K_{W^\nu} + B_{W^\nu}$ is a log divisor with singularities similar to those of $K+B$ near W;
- The moduli part M_{W^ν}, uniquely defined by the adjunction formula

$$(K_X + B)|_{W^\nu} \sim_{\mathbb{R}} K_{W^\nu} + B_{W^\nu} + M_{W^\nu}$$

  is an $\mathbb{R}$-free divisor.

This chapter contains partial results towards this conjecture.

4.1 The different

Definition 4.1. *We say that $j : Y \to (X, B)$ is an* adjunction setting *if the following hold:*

1. *$j : Y \to X$ is a proper morphism of normal varieties, generically one-to-one onto its image $j(Y) = W$;*
2. *$K_X + B$ is a K-Cartier log divisor such that $a(\eta_W; X, B) = 0$.*

Remark 4.1. *If W^ν is the normalization of W, then $\nu : W^\nu \to (X, B)$ is an adjunction setting and there exists a unique birational contraction $\sigma : Y \to W^\nu$ making the following diagram commutative:*

$$\begin{tikzcd} Y \arrow[rr, "\sigma"] \arrow[dr, "j"'] & & W^\nu \arrow[dl, "\nu"] \\ & (X, B) & \end{tikzcd}$$

Conversely, any such birational contraction $\sigma : Y \to W^\nu$ induces the adjunction setting $\nu \circ \sigma : W^\nu \to (X, B)$.

The main property of adjunction settings is that the log divisor $K_X + B$ has a natural *different* $B_Y \in N^1(Y) \otimes K$, measuring the singularities of (X, B) over the codimension 1 points of Y. To define the different, we assume that $j(Y)$ is an exceptional lc center of $K_X + B$. See Remark 4.3 for the general case.

Definition 4.2. *Assume $j : Y \to (X, B)$ is an adjunction setting and $j(Y)$ is an exceptional lc center. Let $\mu : (\tilde{X}, \tilde{B}) \to (X, B)$ be a log resolution such that E, the lc place over $j(Y)$, is realized as a divisor on $\tilde{X}$ and the induced map $E \to j(Y)$ factors through Y:*

$$\begin{tikzcd} (E, B_E) \arrow[r, "\subset"] \arrow[d, "f"'] & (\tilde{X}, \tilde{B}) \arrow[d, "\mu"] \\ Y \arrow[r, "j"] & (X, B) \end{tikzcd}$$

Set $B_E = A|_E$, where $A = \tilde{B} - E$ is a divisor not containing E in its support. Then the different *of the log divisor $K_X + B$ on Y, denoted B_Y, is defined as the discriminant on Y of the log divisor $K_E + B_E$.*

Remark 4.2. *The definition is independent on the choice of the log resolution μ. Indeed, using Hironaka's hut, assume $\mu' = \mu \circ \tau : (\tilde{X}', \tilde{B}') \to (X, B)$ is another log resolution induced by the extraction $\tau : \tilde{X}' \to \tilde{X}$. Let E' and E be the lc places above $j(Y)$ on $\tilde{X}'$ and $\tilde{X}$ respectively. By uniqueness, E' is the proper transform of E via τ and $(\tau|_{E'}) : E' \to E$ is an extraction. In particular, $\tau^*(K_{\tilde{X}} + E + A) = K_{\tilde{X}'} + E' + A'$, so the classical adjunction formula gives*

$$(\tau|_{E'})^*(K_E + B_E) = K_{E'} + B_{E'}$$

Therefore (E, B_E) and $(E', B_{E'})$ have the same discriminant on Y according to Remark 3.1.2.

Lemma 4.1. *In the notations of Definition 4.2, assume moreover that $f(Y)$ is an irreducible component of $LCS(X, B)$. Then the following hold:*

1. *$K_E + B_E \sim_K f^* j^*(K_X + B)$.*
2. *f is a contraction.*
3. *(E, B_E) is Kawamata log terminal over η_Y and $\mathcal{O}_{Y,\eta_Y} = (f_* \mathcal{O}_E(\ulcorner -B_E \urcorner))_{\eta_Y}$.*
4. *$a(Z; X, B) \le a((\mu|_E)^{-1}(Z); E, B_E)$ for every closed subset $Z \subseteq j(Y)$.*

Proof. 1. We have $K_E + B_E \sim_K (K_{\tilde{X}} + \tilde{B})|_E \sim_K \mu^*(K_X + B)|_E = f^* j^*(K_X + B)$.

2. Since $W = j(Y)$ is an irreducible component of $LCS(X, B)$, there exists an open subset $U \subseteq X$ such that $W \cap U$ is the only lc center for $(U, B|_U)$. Since W is an exceptional center, $LCS(\tilde{X}, \tilde{B})|_{\mu^{-1}(U)} = E|_{\mu^{-1}(U)}$, so the the Connectivity Lemma implies that $E \to W$ has connected fibers over $U \cap W \neq \emptyset$. The induced morphism $E \to W^\nu$ has thus connected fibers, so the same holds for $f : E \to Y$.

3. If $U = X \setminus \mu(\mathrm{Supp}(A^{\ge 1}))$ then $W \cap U \neq \emptyset$ and (E, B_E) is Kawamata log terminal over $V = j^{-1}(U)$. The Connectivity Lemma also implies that $\mathcal{O}_Y|_V = f_* \mathcal{O}_E(\ulcorner -B_E \urcorner)|_V$. □

Lemma 4.2. *Let $j : Y \to (X, B)$ be an adjunction setting such that $j(Y)$ is an exceptional component of $LCS(X, B)$. Then the following hold:*

1. *Let D be an $\mathbb{R}$-Cartier divisor on X not containing $j(Y)$ in its support. Then $j : Y \to (X, B + D)$ is an adjunction setting and $(B + D)_Y = B_Y + j^*(D)$.*
2. *Let τ be an extraction and j, j' two adjunction settings making commutative the following diagram:*

$$\begin{tikzcd} Y' \arrow[rr, "\tau"] \arrow[dr, "j'"'] & & Y \arrow[dl, "j"] \\ & (X, B) & \end{tikzcd}$$

   Then $B_Y = \tau_*(B_{Y'})$.

3. *Let $\nu : W^\nu \to (X, B)$ be the induced adjunction setting. Then B_{W^ν} is an effective divisor if B is.*

Proof. (cf. [Ka2, Ka3]) The first two properties are formal consequences of the definition of the residue and Remark 3.1.6. As for the last statement, let $\nu : Y = W^\nu \to (X, B)$ be the induced adjunction setting. By the Kawamata-Viehweg vanishing theorem $R^1\mu_*\mathcal{O}_{\tilde{X}}(-E + \ulcorner -A \urcorner) = 0$, hence the morphism $\mu_*\mathcal{O}_{\tilde{X}}(\ulcorner -A \urcorner) \to \mu_*\mathcal{O}_E(\ulcorner -B_E \urcorner)$ is surjective. Let Q be a prime divisor on Y and assume by contradiction that $b_Q < 0$. Lemma 3.1 shows that $B_E = A|_E$ has coefficients less than 1 over η_Q, hence the same holds for A. Since B is effective, we deduce that $\mu_*\mathcal{O}_{\tilde{X}}(\ulcorner -A \urcorner)_Q = \mathcal{O}_{X,Q}$. Therefore the induced map

$$\mathcal{O}_{Y,Q} \to f_*\mathcal{O}_E(\ulcorner -B_E \urcorner)_Q$$

is surjective. This contradicts the conclusion of Lemma 3.1. Our assumption was false, hence $b_Q \geq 0$. □

Lemma 4.3. *Assume the Base Change Conjecture 5 holds true. Let $j : Y \to (X, B)$ be an adjunction setting such that $j(Y)$ is an exceptional irreducible component of $LCS(X, B)$. Let $\sigma : Y \to W^\nu$ be the induced extraction.*

$$\begin{tikzcd}
(Y, B_Y) \arrow[rr, "\sigma"] \arrow[dr, "j"'] & & (W^\nu, B_{W^\nu}) \arrow[dl, "\nu"] \\
& (X, B) &
\end{tikzcd}$$

Then $K_Y + B_Y$ and $K_{W^\nu} + B_{W^\nu}$ are $\mathbb{Q}$-Cartier log divisors and

$$\sigma^*(K_{W^\nu} + B_{W^\nu}) = K_Y + B_Y$$

Proof. It is a formal consequence of the Base Change Conjecture applied to $f : (E, B_E) \to W^\nu$ under the birational base change σ. □

Remark 4.3. *All the concepts in this chapter are well defined for non-exceptional lc centers too, provided the adjunction calculus on seminormal varieties is developed (representatives on log resolutions for lc places over W are no longer irreducible, but they are simple normal crossings, hence seminormal).*

4.2 Positivity of the moduli part

Let $j : Y \to (X, B)$ be an adjunction setting, E the lc place over $j(Y)$, $f : (E, B_E) \to Y$ the induced morphism and B_Y the induced residue on Y. Assume $K_Y + B_Y$ is $\mathbb{R}$-Cartier. This assumption is satisfied if Y is $\mathbb{Q}$-factorial and it should always hold according to the Base Change Conjecture.

Since $K_E + B_E \sim_{K,f} 0$ and f is a contraction, there exists a unique class $M_Y \in \mathrm{Pic}(Y) \otimes \mathbb{R}$ such that

$$K_E + B_E - f^*(K_Y + B_Y) \sim_{\mathbb{R}} f^*M_Y$$

The class M_Y does not depend on the choice of the realization of E, and it is called the *moduli part* of $K_X + B$ on Y [Ka2, Ka3] . An equivalent definition of M_Y is given by the following *adjunction formula*:

$$j^*(K_X + B) \sim_{\mathbb{R}} K_Y + B_Y + M_Y$$

The appearance of the moduli part was first pointed out by Y. Kawamata. It is trivial in the case $\mathrm{codim}(j(Y), X) = 1$, according to the following result.

Theorem 4.4. *[Sho4] Let $j : Y \to (X, B)$ be an adjunction setting such that $j(Y) = W$ is a prime divisor on X. Let B_Y be the induced residue on Y.*

- *$K_Y + B_Y$ is K-Cartier and $r(K_Y + B_Y)$ is Cartier if $r(K_X + B)$ is Cartier for some $r \in \mathbb{N}$.*
- *If $\sigma : Y \to W^\nu$ is the induced extraction, then $K_Y + B_Y = \sigma^*(K_{W^\nu} + B_{W^\nu})$.*

$$\begin{tikzcd} (Y, B_Y) \arrow[rr, "\sigma"] \arrow[dr, "j"'] & & (W^\nu, B_{W^\nu}) \arrow[dl, "\nu"] \\ & (X, B) & \end{tikzcd}$$

- *The moduli part M_Y is trivial, that is $j^*(K_X + B) \sim_K K_Y + B_Y$.*
- *$a(Z; X, B) \le a(j^{-1}(Z); Y, B_Y)$ for every proper subvariety $Z \subset W$.*

Proof. Since $codim(W, X) = 1$, W is an exceptional lc center and the induced morphism $f : E \to Y$ is an extraction. By Example 3.1.1, $K_Y + B_Y$ is K-Cartier and $f : (E, B_E) \to (Y, B_Y)$ is a crepant extraction. Therefore $\sigma : (Y, B_Y) \to (W^\nu, B_{W^\nu})$ is also crepant. The rest is formal. $\square$

Theorem 4.5. *(cf. [Ka3, Theorem 1]) Let $j : Y \to (X, B)$ be an adjunction setting such that $W = j(Y)$ is an exceptional irreducible component of $LCS(X, B)$. Then j is dominated by an adjunction setting $j' : Y' \to (X, B)$ satisfying the following properties:*

1. *${j'}^*(K_X + B) \sim_{\mathbb{Q}} K_{Y'} + B_{Y'} + M_{Y'}$, where $B_{Y'}$ is the residue of $K_X + B$ on Y' and $M_{Y'} \in Pic(Y') \otimes \mathbb{Q}$ is the moduli part;*
2. *$M_{Y'}$ is nef;*
3. *For any $\mathbb{Q}$-Cartier divisor on X whose support does not contain W the following hold:*
   a) If $(X, B + D)$ is log canonical on an open subset $U \subseteq X$, then $(Y', (B + D)_{Y'})$ is log canonical on ${j'}^{-1}(U)$.
   b) If W is the minimal lc center of $(X, B + D)$ at a closed point $j(y) \in X$, then $(Y', (B + D)_{Y'})$ is Kawamata log terminal on a neighborhood of ${j'}^{-1}(j'(y))$.

Proof. Denote by E the unique lc place over W. There exists an extraction $\sigma : Y' \to Y$ such that $E \to Y$ factors through $f : E \to Y'$ and $f : (E, B_E) \to Y'$ satisfies the property $(\star)$. Therefore the first two properties hold.

From Lemma 4.1.4 and Remark 3.1v), the last part holds for $D = 0$. Finally, let D be a $\mathbb{Q}$-Cartier divisor on X whose support does not contain W. There exists an extraction $\tau : Y'' \to Y'$ such that both $f : (E, B_E) \to Y'$ and $f' : (E', (B + D)_{E'}) \to Y''$ satisfy the property $(\star)$, and moreover f' is induced by the birational morphism $\tau : Y' \to Y$. By the previous step, $(Y'', (B + D)_{Y''})$ has the required properties. Moreover, Proposition 3.6 gives the inequality

$$\tau^*(K_{Y'} + B_{Y'} + {j'}^*(D)) \le K_{Y''} + B_{Y''} + (\tau \circ j')^*(D) = K_{Y''} + (B + D)_{Y''}$$

which implies that these properties are inherited by $(Y', (B + D)_{Y'})$. $\square$

For some applications, the following result of Y. Kawamata suffices.

Theorem 4.6. *[Ka3, Theorem 1] Let (X, B) be a projective log variety and let W be a minimal center of log canonical singularities for (X, B). Assume there exists an effective $\mathbb{Q}$-divisor B^o on X such that $B^o \leq B$ and (X, B) is Kawamata log terminal in a neighborhood of W.*
Let H be an ample Cartier divisor on X, and ϵ a positive rational number. Then there exists an effective $\mathbb{Q}$-divisor D_W on W such that

$$(K_X + B + \epsilon H)|_W \sim_{\mathbb{Q}} K_W + D_W$$

and that the pair (W, D_W) is Kawamata log terminal.

Proof. [Ka3, Theorem 1] We assume $dim W > 0$, otherwise there is nothing to prove. Then $a(\eta_W; X, B) = 0$ and (X, B) is log canonical in a neighborhood of W according to 1.4.1. Moreover, W is normal from the Conectedness Lemma.
There exists an effective $\mathbb{Q}$-divisor D' passing through W such that W is an exceptional minimal lc center of $(X, B^o + (1-t)(B - B^o) + tD')$ for $0 < t \ll 1$. From the previous theorem, there exists a resolution $\sigma : Y \to W$ such that the differents $(B^o + (1-t)(B - B^o) + tD')_Y$ are supported in a simple normal divisor Q for every $0 < t \ll 1$ and

$$M_t = \sigma^*(K_X + B^o + (1-t)(B - B^o) + tD'|_W) - (K_Y + B^o + (1-t)(B - B^o) + tD')_Y)$$

is a nef divisor for every $0 < t \ll 1$. Let $B_Y = \lim_{t \to 0} B^o + (1-t)(B - B^o) + tD'$, which is a divisor supported in Q (this is the different in the non-exceptional case).
Then $M_0 = \sigma^*(K_X + B|_W) - (K_Y + B_Y)$ is a nef divisor. Moreover, B_Y has coefficients less than 1 and is relative effective over W. Let Q' be an effective σ-exceptional $\mathbb{Q}$-divisor with very small coefficients such that $-Q' + M_0 + \sigma^*(\epsilon H)$ is ample on Y, hence there exists an effective $\mathbb{Q}$-divisor M' on Y such that $M' \sim_{\mathbb{Q}} -Q' + M_0 + \sigma^*(\epsilon H)$ and $\mathrm{Supp}(M') \cup \mathrm{Supp}(Q) \cup \mathrm{Supp}(B_Y)$ is a simple normal crossing divisor. Since $K_Y + B_Y + (Q' + M') \sim_{\mathbb{Q},\sigma} 0$, we obtain that $D_W = \sigma_*(B_Y + Q' + M')$ is an effective divisor such that $K_W + D_W$ is $\mathbb{Q}$-Cartier, $\sigma^*(K_W + D_W) = K_Y + (B_Y + Q' + M')$ and $K_W + D_W \sim_{\mathbb{Q}} (K_X + B)|_W$. Finally, (W, D_W) has Kawamata log terminal singularities since it has $(Y, B_Y + Q' + M')$ as a log resolution. ☐

Remark 4.4. *The same proof gives a localized version of the above theorem: if W is the minimal lc center at a closed point $x \in X$ and $dim W > 0$, then we can choose such a divisor D_W such that (W, D_W) is Kawamata log terminal in a neighborhood of x.*

5 Building singularities

Let (X, B) be a log pair, H an ample $\mathbb{Q}$-Cartier divisor and $x \in X \setminus LCS(X, B)$ a closed point. For every $c > 0$ denote by $\mathcal{S}_x(B, H; c)$ the set of all effective $\mathbb{Q}$-Cartier divisors D on X such that

- $D \sim_{\mathbb{Q}} cH$;
- $a(x; B + D) = 0$.

Since H is ample, $\mathcal{S}_x(B, H; c) \neq \emptyset$ for c sufficiently large, so the following infimum is well defined:

$$bld_x(B; H) = \inf\{c > 0; \mathcal{S}_x(B, H; c) \neq \emptyset\}$$

The problem of effective building of singularities consists of finding upper bounds for $bld_x(B; H)$ in terms of the invariants of the local singularity $x \in (X, B)$ and the global numeric properties of H.

Example 5.1. *a) If X is a curve and H is a $\mathbb{Q}$-ample divisor then*

$$bld_x(B; H) = a(x; X, B)/deg_X(H)$$

b) Let x be the vertex of the singular conic $X \subset \mathbb{P}^3$ and let H be the hyperplane section. Then $bld_x(0; H) = a(x; X, 0) = 1$. Note that $H^2 = 2$ and $\inf_{C \subset X}(H \cdot C) = 1$.

Definition 5.1. *a) For a nef divisor $H \in Div(X) \otimes \mathbb{R}$ on a complete variety X we denote by $deg_X(H) = (H^{dim X})_X \in \mathbb{R}$ its top self intersection.*

b) We say that H is normalized at a closed point *$x \in X$ if $deg_W(H|_W) \geq 1$ for every closed subvariety $x \in W \subseteq X$.*

c) (V. V. Shokurov) A divisor H on a variety X has height at least *h if*

$$H \equiv \sum_j h_j H_j + N$$

where H_j are ample Cartier divisors, $h_j > 0\ \forall j$, $\sum_j h_j \geq h$, and N is a nef $\mathbb{R}$-divisor.

Note that $\frac{1}{h}H$ is normalized at every closed point $x \in X$ if H has height at least h.

Inspired by the curve case we expect the following conjecture to hold:

Conjecture 7. *Let (X, B) be a log variety, $x \in X \setminus LCS(X, B)$ a closed point and $H \in Div(X) \otimes \mathbb{Q}$ an ample divisor normalized at x. Then*

$$bld_x(B; H) \leq a(x; X, B)$$

5.1 A first estimate

We use the standard abbreviation $h^0(X,\mathcal{N}) = \dim_k H^0(X,\mathcal{N})$ for the dimension of the space of global sections of a coherent sheaf $\mathcal{N}$ on a variety X.

Definition 5.2. *We say that a morphism of varieties $f : X \to P$ is a generic contraction if it is proper and the induced map $\mathcal{O}_{f(X)} \to f_*\mathcal{O}_X$ is an isomorphism in the generic point of the induced subscheme $f(X) \subset P$.*

The point of the Definition 5.2, formalized in Lemma 5.1, is that if $f : X \to P$ is a generic contraction and $\mathcal{H}$ is an ample line bundle on P, even if not all global sections of $f^*\mathcal{H}$ are pull backs of global sections of $\mathcal{H}$, almost all of them are. For our purposes, X will be the normalization or a resolution of a generically reduced subvariety of P.

Lemma 5.1. *Let $f : X \to P$ be a generic contraction, let $\mathcal{H}$ be an ample line bundle on P, and let*

$$V_k = \mathrm{Im}[H^0(P,\mathcal{H}^{\otimes k}) \to H^0(X, f^*\mathcal{H}^{\otimes k})]$$

Then

$$\lim_{k\to\infty} \frac{h^0(X, f^*\mathcal{H}^{\otimes k})}{\dim V_k} = 1.$$

Proof. Let $\mathcal{G} = \mathrm{Coker}(\mathcal{O}_P \to f_*\mathcal{O}_X)$. The cohomological interpretation of ampleness gives the exactness of the following sequence

$$H^0(P,\mathcal{H}^{\otimes k}) \to H^0(P, f_*\mathcal{O}_X \otimes \mathcal{H}^{\otimes k}) \to H^0(P,\mathcal{G} \otimes \mathcal{H}^{\otimes k}) \to 0$$

for $k \in \mathbb{N}$ large enough. Note that $H^0(X, f^*\mathcal{H}^{\otimes k}) = H^0(P, f_*\mathcal{O}_X \otimes \mathcal{H}^{\otimes k})$ by the projection formula. There exist polynomials $P(k)$ and $Q(k)$, of degrees $\dim \mathrm{Supp}(f_*\mathcal{O}_X)$ and $\dim \mathrm{Supp}(\mathcal{G})$ respectively, such that

$$h^0(P, f_*\mathcal{O}_X \otimes \mathcal{H}^{\otimes k}) = \chi(P, f_*\mathcal{O}_X \otimes \mathcal{H}^{\otimes k}) = P(k),$$

$$h^0(P,\mathcal{G} \otimes \mathcal{H}^{\otimes k}) = \chi(P,\mathcal{G} \otimes \mathcal{H}^{\otimes k}) = Q(k)$$

for $k \in \mathbb{N}$ large enough. Note that $\dim \mathrm{Supp}(f_*\mathcal{O}_X) > \dim \mathrm{Supp}(\mathcal{G})$ since f is a generic contraction. Therefore $\dim V_k = h^0(P, f_*\mathcal{O}_X \otimes \mathcal{H}^{\otimes k}) - h^0(P,\mathcal{G} \otimes \mathcal{H}^{\otimes k})$ is a polynomial in k for k large enough. Moreover, it has the same degree and leading coefficient as $P(k)$, hence the claim. $\square$

Lemma 5.2. *[Sho2, 1.3] Let $f : X \to P$ be a generic contraction of normal varieties and H an ample $\mathbb{Q}$-Cartier divisor on P such that $deg_X(f^*H) > 1$. There exists a natural number $k \in \mathbb{N}$ such that kH is an integer divisor and for any nonsingular point $x \in X$ there exists an effective divisor $D_x \in |kH|$ whose support does not contain $f(X)$ and $\mathrm{mult}_x(f^*D_x) > k$.*

Proof. We use the notation from the previous lemma. Denote also $d = dimX$ and $q^d = deg_X(f^*H)$. We have to prove that the natural map of vector spaces

$$\varphi_k : V_k \to H^0(X, f^*\mathcal{O}(kH) \otimes \mathcal{O}_X/m_x^{kq})$$

has a nontrivial kernel for $k \in \mathbb{N}$ sufficiently large and divisible. Since f^*H is nef and big on X, we have

$$\dim H^0(X, f^*\mathcal{O}(kH)) = \frac{q^d}{d!}k^d + O(k^{d-1})$$

for $k \in \mathbb{N}$ sufficiently large and divisible. From the previous lemma, we deduce that

$$\dim V_k > \frac{q^d}{d!} k^d + O(k^{d-1})$$

for $k \in \mathbb{N}$ large enough. However,

$$\dim H^0(X, f^*\mathcal{O}(kH) \otimes \mathcal{O}_X/m_x^{kq}) = \dim H^0(X, \mathcal{O}_X/m_x^{kq}) = \frac{q^d}{d!} k^d + O(k^{d-1})$$

hence the morphism φ_k cannot be injective for k large enough and divisible. Notice that this k is independent of the choice of the smooth point x. □

5.2 A quadratic bound

Theorem 5.3. *(cf. [Ko2, Theorem 6.7.1]) Let $f : X \to P$ be a generic contraction of normal varieties, $x \in X$ a closed point, and let P be polarized by a $\mathbb{Q}$-ample divisor H such that $deg_X(f^*H) > 1$. Then there exists an effective divisor $D_x \in Div(P) \otimes \mathbb{Q}$ satisfying the following properties:*

1. *$D_x \sim_{\mathbb{Q}} cH$ for some rational number $c < dimX$ and $f(X)$ is not contained in the support of D_x,*
2. *$x \in LCS(X, B + f^*D_x)$ for every effective $\mathbb{R}$-Weil divisor B on X such that $K_X + B$ is $\mathbb{R}$-Cartier.*

Proof. The above statement is stronger than [Ko2, Theorem 6.7.1], but with the same proof, presented here for completeness.
Step 1. Assume first that $x \in X$ is nonsingular. Lemma 5.2 gives a divisor $D'_x \sim kH$ such that B' does not contain $f(X)$ in its support and $mult_x(f^*D'_x) > k$. Then $B_x = \frac{dimX}{k} D'_x$ satisfies the required properties at x. Indeed, if $K_X + B$ is $\mathbb{R}$-Cartier, B is effective and E is the exceptional divisor on the blow-up of X in x, then

$$a(E; B + B_x) = dimX - \mathrm{mult}_x(B) - \frac{\mathrm{mult}_x(f^*D'_x)}{k} dimX < 0.$$

Step 2. We are left with the case when $x \in X$ is a singular point. Since the integer k does not depend on the choice of the smooth point, we can assume there exists a smooth pointed curve $(C, 0)$, a morphism $g : C \to X$ such that $g(0) = x$, $g(C \setminus \{0\}) \subset Reg(X)$ and a morphism $\tilde{g} : C \to |kH|$ such that $B_{g(c)} = \frac{dimX}{k} \tilde{g}(c)$ satisfies the required property at $g(c) \in X$ for each $c \in C \setminus \{0\}$. By Proposition 1.4, $B_x = D_{g(0)}$ also satisfies the required property at $g(0) = x$. □

The following lemma extends Theorem 5.3 to relative effective pseudo-boundaries.

Lemma 5.4. *Let $\sigma : Y \to W$ be a birational contraction, $w \in W$ a closed point, $K_Y + B_Y$ a log divisor and D an effective $\mathbb{R}$-Cartier on W with the following properties:*

1. *$K_Y + B_Y + M \sim_{\mathbb{Q},\sigma} 0$, where $M \in Div(Y) \otimes \mathbb{Q}$ is a nef divisor,*
2. *$\sigma_* B_Y$ is an effective divisor,*

3. $w \in LCS(W, B + D)$ *for every effective* $\mathbb{R}$*-divisor* $B \geq \sigma_* B_V$ *such that* $K_W + B$ *is* $\mathbb{R}$*-Cartier.*

Then $\sigma^{-1}(w) \cap LCS(Y, B_Y + \sigma^* D) \neq \emptyset$*.*

Proof. (cf. [Ka3]) The assumptions are invariant under blow-ups on Y, so we can assume that $S = Exc(\sigma) \cup \mathrm{Supp}(B_Y) \cup \mathrm{Supp}(\sigma^* D)$ has normal crossing. By contradiction, there exists $0 < \epsilon \ll 1$ such that

$$a(E; B_Y + \sigma^* D) \geq \epsilon$$

for every prime divisor E on Y such that $\sigma^{-1}(w) \cap E \neq \emptyset$. Let A be an effective exceptional divisor on Y with coefficients less than ϵ, such that $-A$ is σ-ample and $M - A + \sigma^* M$ is $\mathbb{Q}$-ample for some ample divisor H on W. Let $M' \sim_{\mathbb{Q}} M - A + \sigma^* M$ be an effective divisor with coefficients less than ϵ such that $S \cup Supp(M')$ has normal crossing. In particular, $a(E; B_Y + \sigma^* D + A + M') = a(E; B_Y + \sigma^* D) - \mathrm{mult}(E; A + M') > 0$ for every prime divisor E on Y such that $\sigma^{-1}(w) \cap E \neq \emptyset$. Therefore

$$\sigma^{-1}(w) \cap LCS(Y, B_Y + \sigma^* D + A + M') = \emptyset$$

Let $B_W = \sigma_*(B_Y + A + M')$. Since $K_Y + B_Y + A + M' \sim_{\mathbb{Q},\sigma} 0$, we deduce that $K_W + B_W$ is $\mathbb{R}$-Cartier and

$$K_Y + B_Y + A + M' = \sigma^*(K_W + B_W)$$

Moreover, $B_W \geq \sigma_* B_Y$, hence $\sigma^{-1}(w) \cap LCS(Y, B_Y + \sigma^* D + A + M') \neq \emptyset$. Contradiction. □

Proposition 5.5. *Let* (X, B) *be log variety and* $x \in X$ *a closed point such that* $a(x; X, B) \geq 0$ *and* (X, B) *has a normalized minimal lc center* W *at* x*. Let* H *be a polarization of* X *such that* $deg_W(H|_W) > 1$*. Assume there exists an effective divisor* $B^o < B$ *such that* (X, B^o) *is Kawamata log terminal at* x*.*

Then there exists an effective divisor $B_1 \in Div(X) \otimes \mathbb{Q}$ *such that*

a) $B_1 \sim_{\mathbb{Q}} c_1 H$*,* $c_1 < dim W_1$

b) $a(x; B + B_1) \geq 0$ *and* $(X, B + B_1)$ *has a normalized minimal lc center* W_1 *at* x*.*

c) $W_1 \subset W$ *and* $dim W_1 < dim W$

Proof. Let $Y \to W$ be a resolution as in Theorem 4.5, and let $\sigma^\nu : Y \to W^\nu$ be the induced map to the normalization of W. Since W is normal at x, we can harmlessly say $x \in W^\nu$. Let $B_x \sim_{\mathbb{Q}} cH$ ($c < dim W$) be the divisor obtained by applying Theorem 5.3 to $x \in W^\nu \to (X, H)$. From Lemma 5.4 applied to $Y \to W^\nu$ and $D = B_x$, we deduce that $LCS(Y, B_Y + \sigma^*(B_x|_W)) \cap \sigma^{-1}(x) \neq \emptyset$, that is $K_Y + B_Y + \sigma^*(B_x|_W)$ is not klt near $\sigma^{-1}(x)$.

Let $\alpha = \sup\{t > 0; (X, B + tB_x)$ is log canonical at $x\}$. Theorem 4.53.b) implies that $\alpha \leq 1$. It is clear that $(X, B + \alpha B_x)$ is log canonical at x, with minimal lc center W_1 at x, strictly included in W. After a small perturbation of B_x along H, we can assume that W_1 is normalized too. Therefore $B_1 = \alpha B_x$ and $c_1 = \alpha c$ have the required properties. □

Theorem 5.6. *[Ko2, Theorem 6.4] Let* (X, B) *be a log variety,* $x \in X \setminus LCS(X, B)$ *a closed point and* $H \in Div(X) \otimes \mathbb{Q}$ *an ample divisor normalized at* x*. Then*

$$bld_x(B; H) < \frac{1}{2} dim X (dim X + 1)$$

Proof. Set $W_0 = X$. By Proposition 5.5, there exists an effective $\mathbb{Q}$-Cartier divisor B_1 such that

1. $B_1 \sim_{\mathbb{Q}} c_1 H$, $c_1 + \epsilon < dimW_0$ for some small enough ϵ
2. $(X, B + B_1)$ is log canonical at x with normalized and minimal lc center W_1 at x
3. $dimW_1 < dimW_0$

We repeat the previous step for W_1 and so forth, only that we apply Proposition 5.5 for $(1 + \frac{\epsilon}{m})H$ instead of H, where $m = \frac{dimX(dimX+1)}{2}$. Thus we obtain a sequence of divisors $B_1, B_2, \cdots$ such that

1. $B_j \sim_{\mathbb{Q}} c_j H$
2. $c_1 + \epsilon < dimW_0$ and $c_j < (1 + \frac{\epsilon}{m})dimW_{j-1}$ for every $j \geq 2$
3. $(X, B + \sum_{k=1}^{j} B_k)$ is log canonical at x with normalized and exceptional lc center W_k at x
4. $dimW_{j+1} < dimW_j$ for every $j \geq 0$

We stop this recursive process at some step s for which $W_s = \{x\}$. This definitely happens for some $s \leq dimW_0$, due to property 4 above. We have $\sum_{k=1}^{s} c_k < dimW_0 - \epsilon + (1 + \frac{\epsilon}{m})\sum_{k=2}^{s} dimW_{k-1} \leq \sum_{k=0}^{s} dimW_c \leq m$. Therefore $B_x = \sum_{k=1}^{s} B_k \sim_{\mathbb{Q}} cH$, $c < m$ and $a(x; B + B_x) = 0$. □

5.3 The conjectured optimal bound

We discuss in the section the connection between Conjecture 7 and the Adjunction Conjecture.

Conjecture 8. *Let (X, B) be a log pair which is Kawamata log terminal in a neighborhood of a closed point $x \in X$. Let $H \in Div(X) \otimes \mathbb{Q}$ be an ample divisor normalized at x and fix $0 < \epsilon \ll 1$. Then there exists an effective $\mathbb{Q}$-Cartier divisor D with the following properties:*

1. *$D \sim_{\mathbb{Q}} cH$*
2. *$0 < (1 - \epsilon)c \leq a(x; B) - a(x; B + D)$*
3. *$(X, B + D)$ is maximally log canonical at x*

Lemma 5.7. *Assume the first two properties of the different in the Adjunction Conjecture hold true. Then the two conjectures 8 and 7 are equivalent.*

Proof. We just need to show that Conjecture 8 implies Conjecture 7. We use induction on $dimX$. Fix $0 < \epsilon \ll 1$ and let $D_0 \sim_{\mathbb{Q}} c_0 H$ be a divisor given by Conjecture 8. We may assume that W, the minimal lc center of $(X, B + D_0)$ at x, is normalized. In particular, x is a normal point of W.
If $W = \{x\}$, then $(1 - \epsilon)c_0 \leq a(x; X, B) - a(x; X, B + D_0) = a(x; X, B)$. We hence assume that $dimW > 0$. Let $(B + D_0)_{W^\nu}$ be the different of $K_X + B + D_0$ on W^ν. Then $(W^\nu, (B + D_0)_{W^\nu})$ is klt at x and $H_{W^\nu} = H|_{W^\nu}$ is ample normalized at x. By induction, there exists an effective divisor $D_1 \in Div(X) \otimes \mathbb{Q}$ with the following properties:

1. $D_1 \sim_{\mathbb{Q}} c_1 H$
2. $(1-\epsilon)c_1 \leq a(x; W^\nu, (B+D_0)_{W^\nu})$
3. $a(x; W^\nu, (B+D_0)_{W^\nu} + D_1|_{W^\nu}) = 0$.

By precise inverse of adjunction, we deduce that $a(x; X, B+D_0+D_1) = 0$. Therefore $D = D_0 + D_1 \in \mathcal{S}_x(B, H; c)$, with $c = c_0 + c_1$. But $(1-\epsilon)c = (1-\epsilon)c_0 + (1-\epsilon)c_1 \leq a(x; B) - a(x; B+D_0) + a(x; W^\nu, (B+D_0)_{W^\nu})$. By precise inverse of adjunction again, $a(x; W^\nu, (B+D_0)_{W^\nu}) \leq a(x; B+D_0)$, hence

$$(1-\epsilon)c \leq a(x; B).$$

Letting ϵ tend to 0 we deduce that $bld_x(B; H) \leq a(x; B)$ □

Lemma 5.8. *Conjecture 8 holds true if either $x \in X$ is a nonsingular point, or H is a base point free ample Cartier divisor.*

Proof. We first show that there exists an effective $\mathbb{Q}$-Cartier divisor $D_1 \sim_{\mathbb{Q}} H$ such that

$$\inf_F \operatorname{mult}_F(f^* D_1) \geq 1 - \epsilon$$

where the infimum is taken after all prime divisors F on birational extractions $f : Y \to X$ with $f(F) = \{x\}$. Indeed, since $deg_X(H) > (1-\epsilon)^{dimX}$, Lemma 5.2 gives the required divisor if x is a nonsingular point. If H is a base point free ample Cartier divisor, then we may take D_1 to be any divisor passing through x in the linear system $|H|$.

Let $c > 0$ such that $(X, B + cD_1)$ is maximally log canonical at x. We can assume that the minimal lc center at x is normalized.
We claim that $D = cD_1$ has the desired properties. Let $f : Y \to X$ be an extraction and $F \subset Y$ a prime divisor such that $f(F) = \{x\}$. Then $a(F; B+D) = a(F; B) - c \cdot \operatorname{mult}_F(f^* D_1) \leq a(F; B) - (1-\epsilon)c$. Therefore

$$a(F; B+D) \leq a(F; B) - (1-\epsilon)c$$

for every prime divisor F with center $\{x\}$ on X. Taking infimum after all such $F's$ we get the desired inequality. □

In particular, Conjecture 7 follows from the precise inverse of adjunction if H is a very ample Cartier divisor.

5.4 Global generation of adjoint line bundles

The main application of effective building of isolated log canonical singularities is to the global generation of adjoint line bundles. The key step is the following result of Y. Kawamata.

Proposition 5.9. *[Ka1, Proposition 2.3] Let (X, B) be a log variety, $x \in X \setminus LCS(X, B)$ a closed point and H an ample $\mathbb{Q}$-Cartier divisor. Assume L is a Cartier divisor on X such that*

- $L \equiv K_X + B + hH$;

- $h > bld_x(B;H)$.

Then the sheaf $\mathcal{I}(X,B)\otimes\mathcal{O}_X(L)$ is generated by global sections at x. This means that there exists a global section $s \in H^0(X,\mathcal{O}_X(L))$ such that $s|_{LCS(X,B)} = 0$ and $s(x) \neq 0$.

Proof. Let $D \in \mathcal{S}_x(B;H)$ with $D \sim_{\mathbb{Q}} cH$ and $c < h$. According to Lemma 5.11, we may assume that $\{x\}$ is a normalized lc center, that is $LCS(X, B+D) \cap U = \{x\}$ for some neighborhood U of x. Since $L \equiv K_X + B + D + (h-c)H$ and $(h-c)H$ is ample, the extension of Kawamata-Viehweg vanishing gives the surjection

$$H^0(X,L) \to H^0(LCS(X,B+D), L|_{LCS(X,B+D)}) \to 0$$

But $LCS(X,B+D) = \{x\} \sqcup X'$, where X' is a closed subscheme of X. In particular,

$$H^0(LCS(X,B+D), L|_{LCS(X,B+D)}) = H^0(\{x\}, L|_{\{x\}}) \oplus H^0(X', L|_{X'})$$

The lifting s of the global section $(1,0)$ satisfies the required properties. Clearly, $s(x) \neq 0$, and $s|_{LCS(X,B)} = 0$ since $LCS(X,B)$ is a closed subscheme of X'. $\square$

Lemma 5.10. *For any divisor $D \in \mathcal{S}_x(B,H;c)$ there exists an effective $\mathbb{Q}$-Cartier divisor $D' \sim_{\mathbb{Q}} H$ such that $D_\epsilon = (1-\epsilon)D + \epsilon D' \in \mathcal{S}_x(B,H;(1-\epsilon)c+\epsilon)$ and*

$$LCS(X, B+D_\epsilon) \cap U = \{x\}$$

for some open neighborhood U of x and for all $0 < \epsilon \ll 1$.

Corollary 5.11. *Let (X,B) be a log variety, $x \in X \setminus LCS(X,B)$ a closed point, and $H \in Div(X)\otimes\mathbb{Q}$ an ample divisor normalized at x. Assume L is a Cartier divisor on X such that $L \equiv K_X + B + hH$ for some real number $h > 0$. Then the sheaf $\mathcal{I}(X,B)\otimes\mathcal{O}_X(L)$ is generated by global sections at x if one of the following holds:*

- *[AS, Ko2]* $h > \frac{dimX(dimX+1)}{2}$
- $h > a(x;X,B)$ *if Conjecture 7 holds true.*

Conjecture 9. *(T. Fujita) Let L be an ample line bundle on a projective algebraic variety X. Then the adjoint line bundles $K_X + mL$ are generated by global sections for $m > dimX$.*

Remark 5.1. *Fujita's Conjecture is implied by Conjecture 7. Indeed, set $B = 0$ and $H = L$. Note that $LCS(X,B) = \emptyset$ and H is normalized in any closed point $x \in X$, since H is a Cartier divisor. For any closed point $x \in X$ we have $a(x;X,0) = dimX$, hence Conjecture 7 would imply that L is globally generated at x for $m > dimX$.*

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DEPARTMENT OF MATHEMATICS, THE JOHNS HOPKINS UNIVERSITY, 3400 NORTH CHARLES STREET, BALTIMORE MD 21218
email: ambro@chow.mat.jhu.edu.